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Translation of "Uvlechniye tverdogo tela v trube potokom neszhimayemoy zhidkosti."

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ENTRAINMENT OF SOLIDS IN A PIPE BY A FLOW OF A VISCOUS INCOMPRESSIBLE LIQUID

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INTRODUCTION

Problems of entrainment of solids in pipes by flow of a viscous incompressible liquid arise in the study of motions of various suspensions, in particular, in the flow of blood through narrow capillaries. /238*

The problem of the motion of a sphere entrained by a viscous liquid in a cylindrical tube was solved in [1] in the Stokes approximation when the radius of the sphere is small in comparison with the radius of the tube.

In the present article, we have given a numercial solution of the problem of entrainment of a cylindrical body by a viscous liquid in a circular tube in its precise formulation. The dimensions of the body were assumed to be commensurable with the radius of the tube.

Consider the flow of a viscous incompressible liquid when there are no gravitational forces in an infinite cylindrical tube of radius R under the action of a constant pressure gradient along the axis of the tube. At a certain instant of time, let us place a cylindrical body of radius A and length L on the aixs of the tube (see Fig. 1). It is rather obvious that, after a certain time required for establishment of motion, the body will move along the axis with a constant velocity V. The force acting on the body will be equal to zero and at both infinities the flow will be of the Poiseuille type. In a coordinate system rigidly connected with the walls of the tube, this flow will be unsteady-state. However, /239
in a system of coordinates connected with the body, the flow will be steady-state.

The problem of entrainment of the solid by a viscous incompressible liquid reduces, therefore, to determination of the velocity of the inertial system of coordinates in which the solution of the corresponding boundary problem for the steady-state Navier-Stokes equations yields a force equal to zero acting on the solid.

1. The Equations and Boundary Conditions

The Navier-Stokes equations, which describe a steady-state motion of an incompressible viscous liquid in cylindrical coordinates can be put in the form [2]

$$D\psi = \omega, \quad D\omega = \frac{1}{r^2}\mathrm{Re}\frac{\partial\psi}{\partial x}\omega + \frac{\mathrm{Re}}{2}\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial r}\right), \tag{1}$$

*Numbers in the margin represent pagination in the foreign text.

where D is Stokes' operator

$$D=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2};$$

ψ is the current, Re is the Reynolds number determined from the radius of the tube R, $\mathrm{Re}=RV_{av}/\nu$, the average velocity V_{av} of the flow of the liquid at infinity (in the coordinate system connected with the walls of the tube), and from the viscosity of the liquid ν.

All the quantities in Eq. (1) are dimensionless. Below, we shall write formulas giving the relation between dimensional and dimensionless quantities (the dimensional quantities in these formulas are indicated by bars or by capital letters):

$$\bar{x}=Rx,\qquad \bar{r}=Rr,\qquad A=Ra,$$
$$L=Rl,\qquad \bar{\psi}=(V_{av}/2)R^2\psi.$$

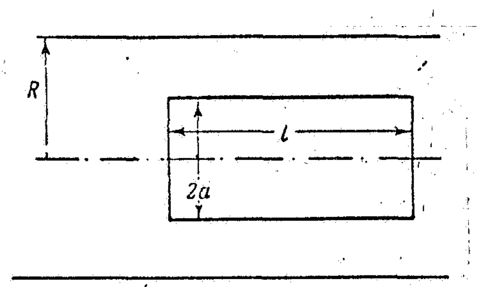

Figure 1.

We shall solve system (1) under boundary conditions listed below. These correspond to the case in which a cylindrical body of radius a and length l moves with a constant velocity V along the axis of a tube within which a liquid is flowing. (The quantity V is dimensionless; the dimensional velocity $\bar{V}$ is connected with the dimensionless velocity by $\bar{V}=(V_{av}/2)V$.) Here, the system of coordinates in which the motion is considered is motionless with respect to the body:

$$\psi=-(1-r^2)^2-0.5\,Vr^2 \quad \text{for } x=\pm\infty; \tag{2}$$

$$\psi=-0.5\,V, \quad \text{when } r=1 \text{ and } -\infty<x<+\infty; \tag{3}$$

$$\psi=-1, \text{ when}$$
$$(r=0)\wedge(x<0\vee x>l)\vee(r=a)\wedge(x>0)\wedge(x<l)\vee(r>0)\wedge$$
$$\wedge(r<a)\wedge(x=0\vee x=l); \tag{4}$$

$$\psi_r=V, \quad \text{when } r=1 \text{ and } -\infty<x<\infty; \tag{5}$$

$$\psi_r=0, \quad \text{when } r=a\wedge x>0\wedge x<l; \tag{6}$$

$$\omega = 0, \quad \text{when } r = 0 \wedge (x < 0 \vee x > l); \tag{7}$$

$$\psi_x = 0, \quad \text{when } r > 0 \wedge r < a \wedge (x = 0 \vee x = l). \tag{8}$$

By using central differences, let us replace the differential equations with differences. We obtain /240

$$\begin{aligned}
\varphi_{i,k} = {} & \frac{\psi_{i,k+1} - 2\psi_{i,k-1} + \psi_{i,k-1}}{h^2} - \frac{1}{kh}\,\frac{\psi_{i,k+1} - \psi_{i,k-1}}{2h} + \\
& + \frac{\psi_{i+1,k} - 2\psi_{i,k} + \psi_{i,k-1}}{H^2} - \omega_{ik} = 0, \\
\Phi_{i,k} = {} & \frac{\omega_{i+1,k} - 2\omega_{i,k} + \omega_{i-1,k}}{H^2} - \frac{1}{kh}\,\frac{\omega_{i,k+1} - \omega_{i,k-1}}{2h} + \\
& + \frac{\omega_{i,k+1} - 2\omega_{i,k} + \omega_{i,k-1}}{h^2} - \mathrm{Re}\,\frac{\omega_{i,k}}{k^2h^2}\,\frac{\psi_{i+1,k} - \psi_{i-1,k}}{2H} - \\
& - \frac{\mathrm{Re}}{8kHh^2}\left[(\psi_{i,k+1} - \psi_{i,k-1})(\omega_{i+1,k} - \omega_{i-1,k}) - (\psi_{i+1,k} - \psi_{i-1,k})(\omega_{i,k+1} - \omega_{i,k-1})\right] = 0,
\end{aligned} \tag{9}$$

where $\psi_{i,k} = \psi(iH, kh)$ and $\omega_{ik} = \omega(iH, kh)$.

Equations (9) hold for all values of i and k for which

$$\begin{aligned}
&((E < i \wedge i < 0 \vee M < i \wedge i < E) \wedge 0 < k \wedge k < G) \vee \\
&\qquad \vee (0 < i \wedge i < M \wedge N < k \wedge k < G),
\end{aligned} \tag{10}$$

where $Nh = a$, $MH = l$, and the quantities E and F are chosen in such a way that their increase will have no effect on the solution obtained.

We need to couple Eqs. (9) with other relations following from the boundary conditions (4)–(8), namely,

$$\psi_{E,k} = \psi_{F,k} = -(1 - k^2h^2)^2 - 0.5\,V^2k^2h^2, \quad \text{when } 0 \leqslant k \wedge k \leqslant G; \tag{11}$$

$$\psi_{i,G} = -0.5\,V, \quad \text{when } E \leqslant i \wedge i \leqslant F; \tag{12}$$

$$\psi_{i,0} = -1, \quad \text{when } E \leqslant i \wedge i \leqslant 0 \vee M \leqslant i \wedge i \leqslant F; \tag{13}$$

$$\psi_{i,N} = -1, \quad \text{when } 0 \leqslant i \wedge i \leqslant M; \tag{14}$$

$$\psi_{0,k}=\psi_{M,k}=-1, \quad \text{when } 0\leqslant k\wedge k\leqslant N; \tag{15}$$

$$\omega_{i,0}=0, \quad \text{when } E\leqslant i\wedge i\leqslant 0\vee M\leqslant i\wedge i\leqslant F; \tag{16}$$

$$\omega_{i,G}=\frac{2}{h^2}(\psi_{i,G+1}-\psi_{i,G}-Vh)+V, \text{ when } E\leqslant i\wedge i\leqslant F; \tag{17}$$

$$\omega_{i,N}=\frac{2}{h^2}(\psi_{i,N-1}+1), \text{ when } 0\leqslant i\wedge i\leqslant M; \tag{18}$$

$$\omega_{0,k}=\frac{2}{H^2}(\psi_{-1,k}+1), \text{ when } 0\leqslant k\wedge k\leqslant N; \tag{19}$$

$$\omega_{M,k}=\frac{2}{H^2}(\psi_{i,k}+1), \text{ when } 0\leqslant k\wedge k\leqslant N. \tag{20}$$

Equations (17)-(20) are derived in a manner analogous to the derivation of the boundary conditions for a vortex in [3-5].

2. The Solution and the Results

Equations (9) and (11)-(20) were solved by the method of successive approximations for a fixed value V_0 of the quantity V. Suppose that $\psi_{i,k}^{(n)}$, $\omega_{i,k}^{(n)}$ is the nth approximation to the values of the functions ψ and ω. Then, let us define the (n + 1)st approximation as follows: first, we calculate the new values of the function ψ from the formulas /241

$$\psi_{i,k}^{(n+1)}=\psi_{i,k}^{(n)}+\alpha\varphi_{i,k}^{(n)}.$$

Then, by using Eqs. (17)-(20), we calculate the new values of the function ω at the boundary points, after which we calculate the new values of the function ω at interior points of the region (that is, for those values of i and k for which the Boolean expression (10) is valid) from the formulas

$$\omega_{i,k}^{(n+1)}=\omega_{i,k}^{(n)}+\beta\Phi_{i,k}^{(n)}.$$

This process continued as long as the following inequality holds:

$$\sum_{i,k}|\varphi_{i,k}|+\sum_{i,k}|\Phi_{i,k}|>10^{-3}, \tag{21}$$

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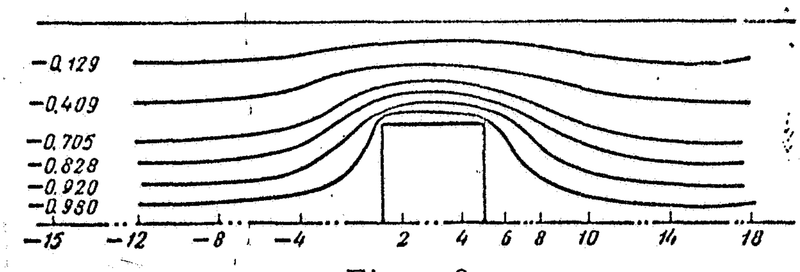

Figure 2.

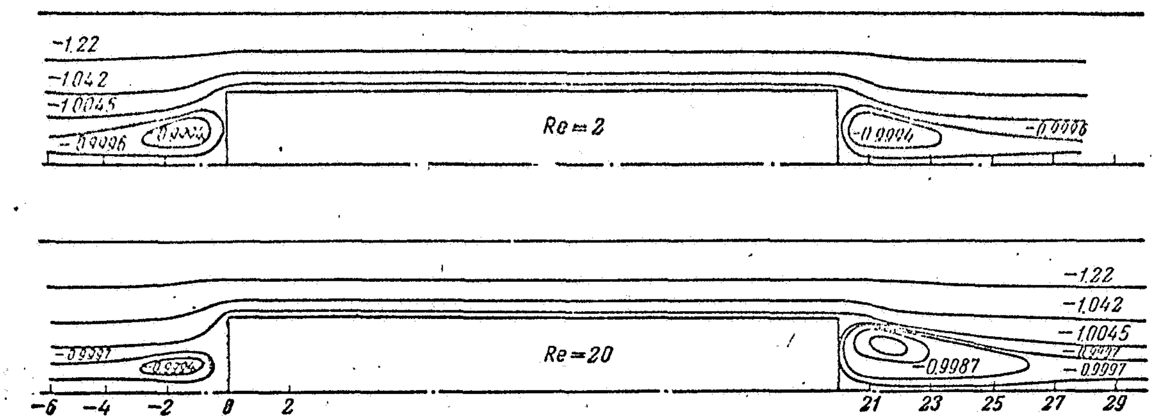

Figure 3.

where the summations are over those values of i and k for which the Boolean expression (10) is valid. With regard to the quantities α and β (time steps), these were chosen in such a way that the fastest convergence of the process is assured. After inequality (21) is no longer satisfied, the problem for the given value of the velocity V_0, was considered solved and the force F_1 acting on the body from the side of the liquid is calculated:

$$\frac{F_1}{\pi\nu\rho R V_{dv}/2} = f = 2\int_0^l \omega(x, a)\,dx - a\int_0^l \omega_r(x, a)\,dx + \int_0^a [\omega_x(0, r) - \omega_x(l, r)]\,r\,dr.$$

Then, a new value of the velocity was calculated from the formula $V_1 = V_0 - \gamma f$.

The process we have been describing continues as long as $\gamma|f| > 10^{-3}$. As soon as $\gamma|f|$ becomes less than 10^{-3}, the problem of the entrainment of the solid by the flow of an incompressible liquid is solved. We denote by V_{entr} the velocity at which $\gamma|f| < 10^{-3}$, and we shall call it the transport velocity.

This computation was carried out on the BESM-6 computer. In the course of the calculation, the values of the parameters E, F, and G were made equal respectively to -16, 64, and 40. The value of the step h in all the calculations was made equal to 0.025. With regard to the step H, the calculations were made mostly with H = 0.2 and 0.1. The values of V_{entr} obtained for H = 0.2 and H = 0.1 differed by less than 0.1%.

Figure 2 shows the field of flow in the case in which the solid is motionless with respect to the walls of the tube and Re = 2 (flow around the solid in the tube). Figure 3 shows the fields of flow in the case in which V = 3.92 (that is, the solid moves along the tube with a velocity just barely less than the maximum velocity of the liquid at infinity) at Reynolds numbers Re = 2 and 20.

Figure 4 shows the field of flow in the case in which the velocity of the solid along the tube is equal to V_{entr} and Re = 40.

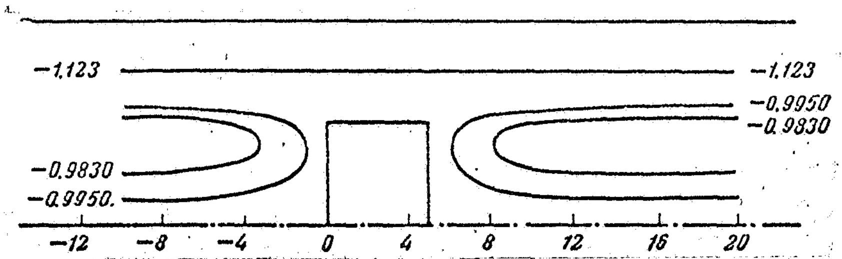

Figure 4.

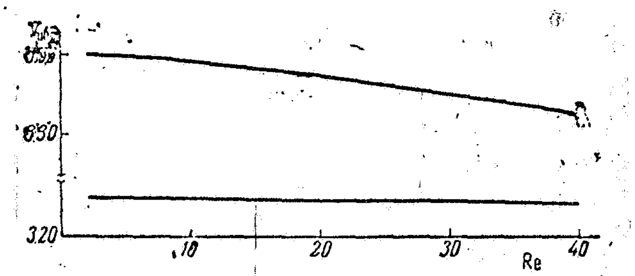

Figure 5.

Figure 5 shows the dependence of the entrainment velocity on Re. The lower curve corresponds to the solid with a = 0.5, $l = 8$ and the upper curve to the solid with a = 0.5, $l = 0.5$.

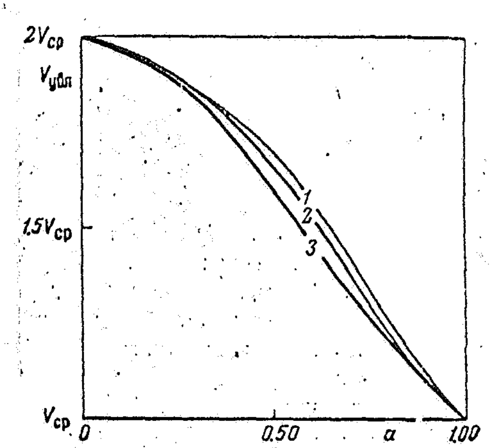

Figure 6.

Figure 6 shows the dependence of the entrainment velocity on the radius of the solid with fixed solid length l and with Reynolds number Re = 2. Curve 1 corresponds to $l = 0.25$, curve 2 to $l = 0.5$, and curve 3 to $l = 4$.

The curves shown in Figures 5 and 6 indicate that in the range of Reynolds numbers at which there is steady-state flow, the entrainment velocity depends only slightly on the Reynolds number or the length of the solid. Obviously, the entrainment velocity is always less than the maximum velocity of the liquid at infinity $V_m = 2V_{av}$ and, as the radius of the solid is increased from 0 to 1, this velocity decreases monotonically from V_m to V_{av}.

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